

Linear dependence and independence

1. You have m vectors $\mathbf{u}_1, \dots, \mathbf{u}_m$ and you create the matrix $U = (\mathbf{u}_1 \cdots \mathbf{u}_m)$. You then

- Find a row-equivalent matrix that is in row-echelon form.
- Determine that that matrix in row-echelon form has m non-zero rows.

Does this mean that every column has exactly one leading non-zero entry?

Does this mean that the rank of the matrix is m ?

Does this mean that the m vectors are linearly independent?

2. In Question 1, suppose that instead, you

- Determine that that matrix in row-echelon form fewer than m non-zero rows.

Does this mean that there are columns that have no leading non-zero entries?

Does this mean that the rank of the matrix is less than m ?

Does this mean that the m vectors are linearly dependent?

Which vectors would you have to remove to create a linearly independent subset of vectors?

3. A laser can only place a positive force on a small object in the direction the laser is being fired, so suppose you put two lasers in a straight line in opposite directions that point at a target in between. Now you can exert a positive force in either direction by choosing one or the other laser and varying the chosen laser's intensity. (See the following paper on laser cooling of atoms:

<https://web.stanford.edu/~rpam/dropoff/Phys041N/lecture6-lasercooling.pdf>.)

These lasers can only exert forces along the vector in which they are pointed. Suppose you have three such pairs of lasers trained on a target, and the vectors of these three lasers are:

$$\begin{pmatrix} 2.1 \\ -4.5 \\ 6.1 \end{pmatrix}, \begin{pmatrix} 1.5 \\ 2.7 \\ -5.3 \end{pmatrix} \text{ and } \begin{pmatrix} -3.6 \\ 1.8 \\ -0.8 \end{pmatrix}.$$

Is it possible to balance a non-zero force applied by one pair of these lasers by appropriate intensities exerted by the other two pairs?

4. Suppose instead in Question 3, the directions of the three lasers are

$$\begin{pmatrix} 2.1 \\ -4.5 \\ 6.1 \end{pmatrix}, \begin{pmatrix} 1.5 \\ 2.7 \\ -5.3 \end{pmatrix} \text{ and } \begin{pmatrix} -4.1 \\ 2.1 \\ -1.2 \end{pmatrix}.$$

Now is it possible to balance a non-zero force applied by one pair of these lasers by appropriate intensities exerted by the other two pairs?